

# A note of the convergence of the Fisher-KPP front centred around its $\alpha$ -level

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## Abstract

We consider the solution  $u(x, t)$  of the Fisher-KPP equation  $\partial_t u = \partial_x^2 u + u - u^2$  centred around its  $\alpha$ -level  $\mu_t^{(\alpha)}$  defined as  $u(\mu_t^{(\alpha)}, t) = \alpha$ . It is well known that for an initial datum that decreases fast enough, then  $u(\mu_t^{(\alpha)} + x, t)$  converges as  $t \rightarrow \infty$  to the critical travelling wave. We study in this paper the speed of this convergence and the asymptotic expansion of  $\mu_t^{(\alpha)}$  for large  $t$ . It is known from Bramson [2] that for initial conditions that decay fast enough, one has  $\mu_t^{(\alpha)} = 2t - (3/2) \ln t + \text{Cste} + o(1)$ . Work is under way [7] to show that the  $o(1)$  in the expansion is in fact a  $k^{(\alpha)}/\sqrt{t} + \mathcal{O}(t^{\epsilon-1})$  for any  $\epsilon > 0$  for some  $k^{(\alpha)}$ , where it is not clear at this point whether  $k^{(\alpha)}$  depends or not on  $\alpha$ . We show that, unless the time derivative of  $\mu_t^{(\alpha)}$  has a very unexpected behaviour at infinity, the coefficient  $k^{(\alpha)}$  does not, in fact, depend on  $\alpha$ .

We also conjecture that, for an initial condition that decays fast enough, one has in fact  $\mu_t^{(\alpha)} = 2t - (3/2) \ln t + \text{Cste} - (3\sqrt{\pi})/\sqrt{t} + g(\ln t)/t + o(1/t)$  for some constant  $g$  which does not depend on  $\alpha$ .

## 1 Introduction

We consider the Fisher-KPP equation [5, 6] which describes the evolution of  $(x, t) \mapsto u(x, t)$ :

$$\partial_t u = \partial_x^2 u + u - u^2, \quad u(x, 0) = u_0(x). \quad (1)$$

Bramson [2] proved that if the initial condition  $u_0(x)$  is such that

$$\begin{cases} 0 \leq u_0(x) \leq 1, \\ \limsup_{x \rightarrow \infty} \frac{1}{x} \ln \left[ \int_x^{x(1+h)} u_0(y) dy \right] \leq -1 \quad \text{for some (all) } h > 0, \\ \lim_{x \rightarrow -\infty} u_0(x) = 1, \end{cases} \quad (2)$$

then the shape of the front around an appropriately chosen centring term  $m_t$  converges to the critical wave  $\omega(x)$ :

$$u(m_t + x, t) \xrightarrow[t \rightarrow \infty]{} \omega(x) \quad \text{uniformly in } x, \quad (3)$$

where  $\omega(x)$  is the unique solution to

$$0 = \omega'' + 2\omega' + \omega - \omega^2, \quad \omega(-\infty) = 1, \quad \omega(+\infty) = 0, \quad \omega(0) = \frac{1}{2}. \quad (4)$$

(The second line of (2) means that  $u_0(x)$  decays roughly as fast or faster than  $e^{-x}$ . The third line could be weakened considerably.)

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Furthermore, if and only if  $u_0(x)$  satisfies the stronger condition

$$\int dx x e^x u_0(x) < \infty, \quad (5)$$

then any valid centring term  $m_t$  in the sense of (3) must be of the form

$$m_t = 2t - \frac{3}{2} \ln t + C + o(1), \quad (6)$$

where the constant  $C$  depends on the initial condition  $u_0(x)$ .

If (2) holds then, as shown in Section 3, for each  $\alpha$  and for each time  $t$  if it is large enough, there exists a unique  $\mu_t^{(\alpha)}$  such that

$$u(\mu_t^{(\alpha)}, t) = \alpha. \quad (7)$$

Furthermore,  $t \mapsto \mu_t^{(\alpha)}$  is  $\mathcal{C}^1$  for  $t$  large enough. Introducing  $W^{(\alpha)}$  as the unique antecedent of  $\alpha$  by  $\omega$ ,

$$\omega(W^{(\alpha)}) = \alpha, \quad (8)$$

it is then easy to see that  $\mu_t^{(\alpha)} - W^{(\alpha)}$  must be a valid choice for  $m_t$  and that one has

$$u(\mu_t^{(\alpha)} + x, t) \xrightarrow[t \rightarrow \infty]{} \omega(W^{(\alpha)} + x) \quad \text{uniformly in } x, \quad (9)$$

In particular, if (5) holds

$$\mu_t^{(\alpha)} = 2t - \frac{3}{2} \ln t + C + W^{(\alpha)} + o(1). \quad (10)$$

where  $C$  is the constant from (6) and, as such, depends on the initial condition but not on  $\alpha$ .

It makes sense to try to determine the next term in the large  $t$  expansion of  $\mu_t^{(\alpha)}$ . A famous conjecture [4] states that

$$\text{If } u_0(x) \text{ decays "fast enough",} \quad \mu_t^{(\alpha)} = 2t - \frac{3}{2} \ln t + C + W^{(\alpha)} - \frac{3\sqrt{\pi}}{\sqrt{t}} + o(t^{-1/2}), \quad (11)$$

where they claim that, remarkably, the coefficient  $-3\sqrt{\pi}$  of the  $t^{-1/2}$  term depends neither on  $\alpha$  nor on the initial condition. Two recent works [3, 1] looking at linearised versions of the Fisher-KPP recover suggest that (11) might only hold if  $\int dx x^2 u_0(x) < \infty$  (compare to the condition (5) under which (10) holds); in particular, if  $u_0(x) \sim Ax^\kappa e^{-x}$  with  $-3 \leq \kappa < -2$ , one would have Bramson's logarithmic correction (10) but the first vanishing correction would be different from that in (11).

Work is under way [7] to prove that, indeed, the first vanishing term in the expansion of  $\mu_t^{(\alpha)}$  is of order  $t^{-1/2}$ , but, as of now, the precise value of the coefficient and, crucially, whether or not it depends on  $\alpha$ , is still an open question.

The goal of this letter is not to prove (11), but to put some constraint on what the first vanishing correction might look like. For instance, our Theorem 1 states that for any initial condition such that (2) holds, and any values of  $\alpha$  and  $\beta$  such that  $0 < \alpha < \beta < 1$ , one has

$$\text{If } 2 - \dot{\mu}_t^{(\alpha)} = \mathcal{O}(t^{-\gamma}) \text{ for some } \gamma > 0, \quad \text{then } \mu_t^{(\alpha)} - \mu_t^{(\beta)} = W^{(\alpha)} - W^{(\beta)} + \mathcal{O}(t^{-\gamma}), \quad (12)$$

where  $\dot{\mu}_t^{(\alpha)}$  is the derivative of  $t \rightarrow \mu_t^{(\alpha)}$ .

A natural question is of course how large can  $\gamma$  be chosen in (12). In the Physics literature, it is often assumed, and without batting an eye, that when (10) holds, then  $2 - \dot{\mu}_t^{(\alpha)}$  must be equivalent to  $3/(2t)$ . Of course, one is not allowed in general to differentiate asymptotic expansions but, intuitively, if the initial condition decays fast enough at infinity, then the heat operator in the Fisher-KPP equation (1) should smooth everything and the functions  $\mu_t^{(\alpha)}$  should be extremely well behaved for large times. It would then seem that " $2 - \dot{\mu}_t^{(\alpha)} = \mathcal{O}(t^{-1})$ " is a fair conjecture. To our knowledge, there is no rigorous result on this, but it would imply the following Conjecture:

**Conjecture 1.** *Pick  $\alpha$  and  $\beta$  in  $(0, 1)$ . For any initial condition  $u_0(x)$  such that (2) holds, then*

$$\mu_t^{(\alpha)} - \mu_t^{(\beta)} = W^{(\alpha)} - W^{(\beta)} + \mathcal{O}\left(\frac{1}{t}\right). \quad (13)$$

In Section 4, we present some numerical evidence in support of Conjecture 1 for a step initial condition. What (13) basically means is that any term larger than  $1/t$  in a large  $t$  asymptotic expansion of  $\mu_t^{(\alpha)}$  must have a coefficient independent of  $\alpha$ . In particular:

- If, as claimed in [4, 7], the first vanishing correction in  $\mu_t^{(\alpha)}$  is of order  $1/\sqrt{t}$  for initial conditions that decay fast enough, then its coefficient must be independent of  $\alpha$ .
- The results of [3, 1] suggest that if the initial condition is asymptotically of the form  $u_0(x) \sim Ax^\kappa e^{-x}$  with  $\kappa \in (-3, -2)$ , then the first vanishing correction should be of order  $t^{1+\frac{\kappa}{2}}$ . If it is indeed the case, our conjecture implies that the coefficient is independent of  $\alpha$ .

If Conjecture 1 turns out to be incorrect and if, for instance, for some initial condition  $u_0(x)$ , one has  $\mu_t^{(\alpha)} = 2t - \frac{3}{2} \ln t + C + W^{(\alpha)} + k^{(\alpha)} t^{-1/2} + \mathcal{O}(t^{-0.99})$  with a coefficient  $k^{(\alpha)}$  which is not a constant function of  $\alpha$ , then Theorem 1 below implies that  $2 - \dot{\mu}_t^{(\alpha)}$  is not a  $\mathcal{O}(1/t)$  and Theorem 3 below implies that  $\dot{\mu}_t^{(\alpha)}$  oscillates around 2 at infinity, which would be quite unexpected.

In Section 5, we present some work on a solvable model in the Fisher-KPP class which was introduced in [3]. This leads us to another Conjecture on the asymptotic expansion of  $\mu_t^{(\alpha)}$

**Conjecture 2.** *For an initial condition  $0 \leq u_0(x) \leq 1$  with  $\lim_{x \rightarrow -\infty} u_0(x) = 1$  and*

$$\int u_0(x) x^3 e^x dx < \infty, \quad (14)$$

*one has*

$$\mu_t^{(\alpha)} = 2t - \frac{3}{2} \ln t + C + W^{(\alpha)} - \frac{3\sqrt{\pi}}{\sqrt{t}} + g \frac{\ln t}{t} + \mathcal{O}\left(\frac{1}{t}\right), \quad (15)$$

*for some constant  $g$  which, by Conjecture 1, does not depend on  $\alpha$ .*

The work presented in Section 5 suggests also that, maybe,  $g = \frac{9}{8}(5 - 6 \ln 2) \approx 0.946$ , and this value is compatible with numerical simulations. However, this value for  $g$  relies on transposing by analogy a result derived on a front equation which is quite different from the Fisher-KPP equation, and it remains of a very speculative nature.

## 2 Results

We restrict ourselves to initial conditions  $u_0(x)$  such that (2) holds. Pick  $\alpha \in (0, 1)$  and introduce

$$\eta_t = 2 - \dot{\mu}_t^{(\alpha)}. \quad (16)$$

Implicitly,  $\eta_t$  depends on  $\alpha$ . One has, for large time [6, 8],

$$\eta_t \rightarrow 0. \quad (17)$$

**Theorem 1.** *Pick  $\alpha \in (0, 1)$ . For any initial condition  $u_0(x)$  such that (2) holds, if*

$$\eta_t := 2 - \dot{\mu}_t^{(\alpha)} = \mathcal{O}(t^{-\gamma}) \text{ for some } \gamma > 0, \quad (18)$$

*then, for any  $x_0 > 0$ ,*

$$\max_{x \in [-x_0, 0]} \left| u(\mu_t^{(\alpha)} + x, t) - \omega(W^{(\alpha)} + x) \right| = \mathcal{O}(t^{-\gamma}), \quad (19)$$

*which implies that for any  $\beta \in (\alpha, 1)$ ,*

$$\mu_t^{(\alpha)} - \mu_t^{(\beta)} = W^{(\alpha)} - W^{(\beta)} + \mathcal{O}(t^{-\gamma}). \quad (20)$$

*If, furthermore,  $\alpha > \frac{1}{2}$ , then the “ $\max_{x \in [-x_0, 0]}$ ” in (19) can be replaced by a “ $\max_{x \leq 0}$ ”.*

**Theorem 2.** Pick  $\alpha \in (0, 1)$ . For any initial condition  $u_0(x)$  such that (2) holds, if

$$\eta_t := 2 - \dot{\mu}_t^{(\alpha)} \neq 0 \text{ for } t \text{ large enough} \quad \text{and} \quad \frac{\dot{\eta}_t}{\eta_t} \rightarrow 0, \quad (21)$$

then, for any  $x_0 > 0$ ,

$$\max_{x \in [-x_0, 0]} \left| \frac{u(\mu_t^{(\alpha)} + x, t) - \omega(W^{(\alpha)} + x)}{\eta_t} - \left( \Phi(W^{(\alpha)} + x) - \frac{\Phi(W^{(\alpha)})}{\omega'(W^{(\alpha)})} \omega'(W^{(\alpha)} + x) \right) \right| \xrightarrow{t \rightarrow \infty} 0, \quad (22)$$

where  $\Phi$  is

$$\Phi(x) = \omega'(x) \int_0^x dy \frac{e^{-2y}}{\omega'(y)^2} \int_{-\infty}^y dz \omega'(z)^2 e^{2z}. \quad (23)$$

This implies that for any  $\beta \in (\alpha, 1)$ ,

$$\mu_t^{(\alpha)} - \mu_t^{(\beta)} = W^{(\alpha)} - W^{(\beta)} - \eta_t \left[ \frac{\Phi(W^{(\alpha)})}{\omega'(W^{(\alpha)})} - \frac{\Phi(W^{(\beta)})}{\omega'(W^{(\beta)})} + o(1) \right]. \quad (24)$$

If, furthermore,  $\alpha > \frac{1}{2}$ , then the “ $\max_{x \in [-x_0, 0]}$ ” in (22) can be replaced by a “ $\max_{x \leq 0}$ ”.

**Theorem 3.** Pick  $\alpha \in (0, 1)$ . For any initial condition  $u_0(x)$  such that (2) holds, if

$$\begin{cases} \mu_t^{(\alpha)} = 2t - \frac{3}{2} \ln t + C + W^{(\alpha)} - \frac{g}{\sqrt{t}} + \mathcal{O}(t^{-\gamma}) & \text{for some } \gamma \in \left(\frac{1}{2}, 1\right], \\ \eta_t := 2 - \dot{\mu}_t^{(\alpha)} \text{ has a constant sign for } t \text{ large enough,} \end{cases} \quad (25)$$

then, for any  $x_0 > 0$ ,

$$\max_{x \in [-x_0, 0]} \left| u(\mu_t^{(\alpha)} + x, t) - \omega(W^{(\alpha)} + x) \right| = \mathcal{O}(t^{-\gamma}), \quad (26)$$

which implies that for any  $\beta \in (\alpha, 1)$ ,

$$\mu_t^{(\beta)} = 2t - \frac{3}{2} \ln t + C + W^{(\beta)} - \frac{g}{\sqrt{t}} + \mathcal{O}(t^{-\gamma}), \quad (27)$$

where we emphasize that the coefficient  $g$  is the same as in (25).

If, furthermore,  $\alpha > \frac{1}{2}$ , then the “ $\max_{x \in [-x_0, 0]}$ ” in (26) can be replaced by a “ $\max_{x \leq 0}$ ”.

*Remarks:*

- Theorem 2 is more precise than Theorem 1, but requires to make some assumptions on the second derivative on  $\mu_t^{(\alpha)}$ .
- In Theorem 2, if  $2 - \dot{\mu}_t^{(\beta)}$  satisfies the same hypothesis as  $\eta_t = 2 - \dot{\mu}_t^{(\alpha)}$ , then it is easy to see that necessarily  $2 - \dot{\mu}_t^{(\beta)} \sim 2 - \dot{\mu}_t^{(\alpha)}$ .
- In Theorem 2, one checks that  $\Phi$  is the unique solution to

$$\Phi'' + 2\Phi' + (1 - 2\omega)\Phi = \omega'(x), \quad \Phi(0) = 0, \quad \Phi(-\infty) = 0. \quad (28)$$

- The results above concern only convergence for negative  $x$ . However, in each Theorem, we could replace the “ $\max_{x \in [-x_0, 0]}$ ” by a “ $\max_{x \leq x_0}$ ” if one assume that the hypothesis on  $\mu_t^{(\alpha)}$  does not hold only for the one value of  $\alpha$  that we pick, but holds in fact for all values of  $\alpha$  (as in “there exists a  $\gamma$  such that (18) or (25) hold for all  $\alpha$ ”, or “(21) holds for all  $\alpha$ ”). Indeed, one would simply have to apply the Theorems as written above once for an  $\alpha'$  small enough to encompass what happens at  $x = x_0$ , another time for an  $\alpha''$  larger than  $1/2$ , and then glue together the results.
- The theorems do not assume that  $u_0(x)$  is such that we are in the regime (5) with the  $-\frac{3}{2} \ln t$  of Bramson. It merely assumes that the critical travelling wave  $\omega$  is reached.
- With Theorem 1, it would be sufficient to prove that  $2 - \dot{\mu}_t^{(\alpha)} = \mathcal{O}(t^{-1})$  for any  $\alpha \in (0, 1)$  to obtain Conjecture 1.

### 3 Proofs

We start by proving the following result which was mentioned in the introduction

**Lemma 1.** *Suppose that the initial condition  $u_0(x)$  is such that (2) holds and fix  $\alpha \in (0, 1)$ . Then, for  $t$  large enough,  $\mu_t^{(\alpha)}$  is the unique solution of  $u(x, t) = \alpha$  and furthermore  $t \mapsto \mu_t^{(\alpha)}$  is differentiable.*

*Proof.* Recall (3): there exists  $m_t$  such that,

$$u(m_t + x, t) \rightarrow \omega(x) \quad \text{uniformly in } x. \quad (29)$$

A standard result (see for instance [8, Theorem 9.1]) gives then that

$$\partial_x u(m_t + x, t) \rightarrow \omega'(x) \quad \text{locally uniformly in } x. \quad (30)$$

For any  $t > 0$  the function  $x \mapsto u(x, t)$  is continuous and interpolates between 1 and 0 so for each  $t$  there exists at least one  $x$  such that  $u(m_t + x, t) = \alpha$ . For each  $\epsilon > 0$ , if time is large enough, then  $u(m_t + x, t) = \alpha$  implies that  $|x - W^{(\alpha)}| \leq \epsilon$  because of the uniform convergence (29). As  $\omega'(x)$  is negative and bounded away from 0 on  $[W^{(\alpha)} - \epsilon, W^{(\alpha)} + \epsilon]$  then  $\partial_x u(m_t + x, t)$  is negative on the same interval for  $t$  large enough because of (30). This implies that for  $t$  large enough there exists a unique  $x$  such that  $u(m_t + x, t) = \alpha$  or, equivalently, a unique  $\mu_t^{(\alpha)}$  such that  $u(\mu_t^{(\alpha)}, t) = \alpha$ . The differentiability of  $t \mapsto \mu_t^{(\alpha)}$  is then a consequence of the implicit function Theorem.  $\square$

We now turn to the proofs of the Theorems. Pick  $\alpha \in (0, 1)$  and an initial condition  $u_0(x)$  satisfying (2). Introduce

$$\tilde{\omega}(x) = \omega(W^{(\alpha)} + x). \quad (31)$$

When  $t$  is sufficiently large so that  $t \mapsto \mu_t^{(\alpha)}$  is a well-defined  $C^1$  function, introduce also

$$\delta(x, t) = u(\mu_t^{(\alpha)} + x, t) - \tilde{\omega}(x). \quad (32)$$

Of course,

$$|\delta(x, t)| \leq 1, \quad \delta(0, t) = 0, \quad \delta(x, t) \xrightarrow[t \rightarrow \infty]{} 0, \quad \text{uniformly in } x. \quad (33)$$

From (32),

$$\begin{aligned} \partial_t \delta &= \partial_x^2 (\delta + \tilde{\omega}) + \dot{\mu}_t^{(\alpha)} \partial_x (\delta + \tilde{\omega}) + (\delta + \tilde{\omega}) - (\delta + \tilde{\omega})^2, \\ &= \partial_x^2 \delta + \dot{\mu}_t^{(\alpha)} \partial_x \delta + (1 - 2\tilde{\omega})\delta - \delta^2 - (2 - \dot{\mu}_t^{(\alpha)})\tilde{\omega}', \\ &= \partial_x^2 \delta + 2\partial_x \delta - (2\tilde{\omega} - 1 + \delta)\delta - \eta_t \tilde{\omega}' - \eta_t \partial_x \delta, \end{aligned} \quad (34)$$

where we used (4) to simplify the  $\tilde{\omega}$  and where we recall (16):

$$\eta_t = 2 - \dot{\mu}_t^{(\alpha)}. \quad (35)$$

Define  $r$  by

$$r(x, t) = e^x \delta(x, t). \quad (36)$$

One finds that  $r$  satisfies for all  $x \in \mathbb{R}$

$$\partial_t r = \partial_x^2 r - (2\tilde{\omega} + \delta)r + (r - \tilde{\omega}'e^x)\eta_t - \eta_t \partial_x r, \quad r(0, t) = 0. \quad (37)$$

The condition  $r(0, t) = 0$  effectively decouples the domains  $x \leq 0$  and  $x \geq 0$ . We can therefore consider (37) for  $x \leq 0$  only. A key step in our proofs is the following:

**Proposition 1.** *With  $u_0(x)$  satisfying (2), there exists two positive constants  $c$  and  $t_0$  such that*

$$\max_{x \leq 0} |r(x, t)| \leq e^{-\alpha t} \left( e^{\alpha t_0} + c \int_{t_0}^t du |\eta_u| e^{\alpha u} \right) \quad \text{for all } t \geq t_0. \quad (38)$$

Furthermore, if  $\alpha > 1/2$ , there exists two other positive constants  $c$  and  $t_0$  such that

$$\max_{x \leq 0} |\delta(x, t)| \leq e^{-(\alpha - \frac{1}{2})t} \left( e^{(\alpha - \frac{1}{2})t_0} + c \int_{t_0}^t du |\eta_u| e^{(\alpha - \frac{1}{2})u} \right) \quad \text{for all } t \geq t_0. \quad (39)$$

The right-hand-sides in (38) and (39) can then be estimated with the following Lemma:

**Lemma 2.** For  $\beta > 0$  and  $t_0$  two real numbers, and  $t \mapsto \phi_t$  a function, define  $R_t$  by

$$R_t = e^{-\beta t} \int_{t_0}^t du \varphi_u e^{\beta u}. \quad (40)$$

For large time,

- If  $\varphi_t \rightarrow 0$ , then  $R_t \rightarrow 0$ ,
- If  $\varphi_t = \mathcal{O}(t^{-\gamma})$  for some  $\gamma > 0$ , then  $R_t = \mathcal{O}(t^{-\gamma})$ ,
- If  $\int_t^\infty \varphi_u du = \mathcal{O}(t^{-\gamma})$  for some  $\gamma > 0$ , then  $R_t = \mathcal{O}(t^{-\gamma})$ .

With Proposition 1 and Lemma 2, the first part of Theorem 1 is trivial: assuming that  $\eta_t = \mathcal{O}(t^{-\gamma})$  with  $\gamma > 0$ , then

$$\begin{cases} \max_{x \in [-x_0, 0]} |\delta(x, t)| \leq e^{x_0} \max_{x \leq 0} |r(x, t)| = \mathcal{O}(t^{-\gamma}), \\ \max_{x \leq 0} |\delta(x, t)| = \mathcal{O}(t^{-\gamma}), \end{cases} \quad \text{if } \alpha > \frac{1}{2}. \quad (41)$$

which is (19) of Theorem 1.

The first part of Theorem 3 is also very easy. Assuming (25), then one has, for some  $\gamma \in (\frac{1}{2}, 1]$ ,

$$\eta_t = \frac{3}{2t} - \frac{g}{2t^{3/2}} + \psi_t \quad \text{with} \quad \int_t^\infty du \psi_u = \mathcal{O}(t^{-\gamma}). \quad (42)$$

Because we assume that  $\eta_t$  does not change sign for  $t$  large enough, one can push the absolute values around  $\eta_t$  in (38) and (39) outside the integral. Then, applying Lemma 2 to each of the three terms composing  $\eta_t$  in (42), one reaches again the conclusion (41) which is (26) in Theorem 3.

The second parts of Theorems 1 and 3 are then direct consequences of their first parts. Pick  $\beta \in (\alpha, 1)$ . By definition,

$$\beta = u(\mu_t^{(\alpha)} + (\mu_t^{(\beta)} - \mu_t^{(\alpha)}), t) = \omega(W^{(\alpha)} + \mu_t^{(\beta)} - \mu_t^{(\alpha)}) + \delta(\mu_t^{(\beta)} - \mu_t^{(\alpha)}, t). \quad (43)$$

We know that  $\mu_t^{(\beta)} - \mu_t^{(\alpha)}$  converges to  $W^{(\beta)} - W^{(\alpha)} < 0$ , so it must remain inside  $[-x_0, 0]$  for  $t$  large enough and a well chosen  $x_0$ . Then, the term  $\delta(\cdot, t)$  in (43) is a  $\mathcal{O}(t^{-\gamma})$  and because  $\omega$  is differentiable with non-zero derivative, one must have

$$\mu_t^{(\beta)} - \mu_t^{(\alpha)} = W^{(\beta)} - W^{(\alpha)} + \mathcal{O}(t^{-\gamma}), \quad (44)$$

which is the second part (20) of Theorem 1. The conclusion (44) also holds for Theorem 3; combined with its hypothesis (25), it gives the second part (27) of Theorem 3.

Therefore, it only remains to prove Proposition 1 and Lemma 2 to complete the proofs of Theorems 1 and 3.

*Proof of Proposition 1.* Recall equation (37) followed by  $r$ ,

$$\partial_t r = \partial_x^2 r - (2\tilde{\omega} + \delta)r + (r - \tilde{\omega}'e^x)\eta_t - \eta_t \partial_x r, \quad r(0, t) = 0. \quad (45)$$

We only consider the side  $x \leq 0$ . Since  $r(x, t) = e^x \delta(x, t)$  we have

$$|r(x, t)| \leq 1 \quad \text{for all } t \text{ and all } x \leq 0. \quad (46)$$

Furthermore,  $-\tilde{\omega}'(x)e^x > 0$  is bounded for  $x \leq 0$  so there exists a  $c$  such that

$$|r(x, t) - \tilde{\omega}'(x)e^x| \leq c \quad \text{for all } t \text{ and all } x \leq 0. \quad (47)$$

Also, there exists a  $t_0$  such that

$$2\tilde{\omega}(x) + \delta(x, t) > \alpha \quad \text{for } t \geq t_0 \text{ and } x \leq 0. \quad (48)$$

Indeed,  $\tilde{\omega}(x) \geq \alpha$  for  $x \leq 0$ , and  $\delta(x, t)$  converges uniformly to 0.

With these ingredients we are ready to apply the comparison principle. It goes in two steps; first, because of (46) and (47) one has for all  $x \leq 0$  and all  $t \geq t_0$

$$r(x, t) \leq \hat{r}(x, t) \quad \text{where} \quad \partial_t \hat{r} = \partial_x^2 \hat{r} - (2\tilde{\omega} + \delta)\hat{r} + c|\eta_t| - \eta_t \partial_x \hat{r}, \quad \hat{r}(x, t_0) = 1, \quad \hat{r}(0, t) = 0. \quad (49)$$

Clearly,  $\hat{r}$  cannot become negative. Then, one gets with (48) that for any non-negative function  $b_t$

$$\hat{r}(x, t) \leq \bar{r}(x, t) \quad \text{where} \quad \partial_t \bar{r} = \partial_x^2 \bar{r} - \alpha \bar{r} + c|\eta_t| - \eta_t \partial_x \bar{r}, \quad \bar{r}(x, t_0) = 1, \quad \bar{r}(0, t) = b_t. \quad (50)$$

We choose  $b_t$  so that  $\bar{r}$  remains  $x$  independent, which leads to  $\partial_t \bar{r} = -\alpha \bar{r} + c|\eta_t|$  or

$$\bar{r}(\cdot, t) = e^{-\alpha t} \left( e^{\alpha t_0} + c \int_{t_0}^t du |\eta_u| e^{\alpha u} \right). \quad (51)$$

Similarly, one shows that  $-r(x, t) \leq \hat{r}(x, t) \leq \bar{r}(\cdot, t)$ , which concludes the proof.

Finally, we prove the second part of Proposition 1 in exactly the same way than the first part, but starting from (34) instead of (37). As above, one first shows that  $\delta \leq \hat{\delta}$  where  $\hat{\delta}$  follows the same equation as  $\delta$  but with the  $-\tilde{\omega}'\eta_t$  replaced by  $c|\eta_t|$ . Then,  $\hat{\delta} \leq \bar{\delta}$  where we replace  $-(2\tilde{\omega} - 1 + \delta)\delta$  by  $-(\alpha - \frac{1}{2})\delta$ . Indeed, for all  $x < 0$ , one has  $2\tilde{\omega}(x) - 1 \geq 2\alpha - 1$  and for  $t$  large enough  $|\delta| \leq \alpha - \frac{1}{2}$ .  $\square$

*Proof of Lemma 2.*

- The first bullet point is easy. Assume  $\varphi_t \rightarrow 0$ . For any  $\epsilon > 0$  pick  $t_1 > t_0$  such that  $|\varphi_t| < \epsilon$  for  $t \geq t_1$ . Then

$$|R_t| \leq e^{-\beta t} \left( \int_{t_0}^{t_1} du |\varphi_u| e^{\beta u} \right) + \epsilon e^{-\beta t} \int_{t_1}^t du e^{\beta u} \leq \frac{2\epsilon}{\beta} \quad \text{for } t \text{ large enough.} \quad (52)$$

- For the second bullet point we prove a slightly more general result. Let  $t \mapsto \tilde{\varphi}_t$  be a function such that

$$\tilde{\varphi}_t > 0, \quad \ln \tilde{\varphi}_t \text{ is convex for } t > t_0, \quad \liminf_{t \rightarrow \infty} \frac{\ln \tilde{\varphi}_t}{t} > -\beta. \quad (53)$$

By convexity, for  $t > t_0$  and  $u \in [t_0, t]$ , one has

$$\ln \tilde{\varphi}_u \leq \frac{\ln \tilde{\varphi}_t - \ln \tilde{\varphi}_{t_0}}{t - t_0} (u - t_0) + \ln \tilde{\varphi}_{t_0} \quad \text{and then} \quad \tilde{\varphi}_u e^{\beta u} \leq \tilde{\varphi}_{t_0} e^{\beta t_0 + \left[ \beta + \frac{\ln \tilde{\varphi}_t - \ln \tilde{\varphi}_{t_0}}{t - t_0} \right] (u - t_0)}. \quad (54)$$

Because of the last hypothesis on  $\tilde{\varphi}$  in (53), there exists a  $c > 0$  such that the term in square brackets in the equation above is larger than  $c$  for  $t$  large enough. Then, for  $t$  large enough,

$$0 \leq \int_{t_0}^t du \tilde{\varphi}_u e^{\beta u} \leq \frac{\tilde{\varphi}_t e^{\beta t} - \tilde{\varphi}_{t_0} e^{\beta t_0}}{c} \quad \text{and then} \quad e^{-\beta t} \int_{t_0}^t du \tilde{\varphi}_u e^{\beta u} = \mathcal{O}(\tilde{\varphi}_t). \quad (55)$$

If one assumes now that  $\varphi_t = \mathcal{O}(\tilde{\varphi}_t)$  where  $\tilde{\varphi}_t$  satisfies (53), then we conclude that  $R_t = \mathcal{O}(\tilde{\varphi}_t)$ . As the functions  $\tilde{\varphi}_t = t^{-\gamma}$  with  $\gamma > 0$  satisfy these conditions, we have proved the second bullet point.

- We finally turn to the third bullet point. Let  $\Phi_t = \int_t^\infty du \varphi_u$ . By integration by parts

$$R_t = \Phi_{t_0} e^{-\beta(t-t_0)} - \Phi_t + \beta e^{-\beta t} \int_{t_0}^t du \Phi_u e^{\beta u} \quad (56)$$

If one assumes that  $\Phi_t = \mathcal{O}(t^{-\gamma})$  for some  $\gamma > 0$ , then an application of the second bullet point gives the third bullet point.  $\square$

We now turn to the proof of Theorem 2.

*Proof of Theorem 2.* Write

$$r(x, t) = \eta_t [\Psi(x) + s(x, t)]. \quad (57)$$

Then, by substituting into (37) and after division by  $\eta_t$ ,

$$\frac{\dot{\eta}_t}{\eta_t} [\Psi + s] + \partial_t s = \Psi'' + \partial_x^2 s - (2\tilde{\omega} + \delta - \eta_t) [\Psi + s] - \tilde{\omega}' e^x - \eta_t [\Psi' + \partial_x s]. \quad (58)$$

We choose for  $\Psi$  the unique solution to

$$\Psi'' - 2\tilde{\omega}\Psi = \tilde{\omega}' e^x, \quad \Psi(0) = 0, \quad \Psi(x) \text{ is bounded for } x < 0. \quad (59)$$

Before going further, let us check that the solution to (59) exists and is unique. First notice that

$$(\tilde{\omega}' e^x)'' - 2\tilde{\omega}(\tilde{\omega}' e^x) = 0. \quad (60)$$

This leads to look for a solution  $\Psi$  of the form

$$\Psi(x) = \tilde{\omega}'(x) e^x \lambda(x). \quad (61)$$

One obtains

$$(\tilde{\omega}' e^x) \lambda'' + 2(\tilde{\omega}' e^x)' \lambda' = \tilde{\omega}' e^x \quad \text{which is the same as} \quad \frac{d}{dx} [(\tilde{\omega}' e^x)^2 \lambda'] = (\tilde{\omega}' e^x)^2. \quad (62)$$

One sees from (60) that  $\tilde{\omega}'(x) e^x \sim C e^{\sqrt{2}x}$  as  $x \rightarrow -\infty$  for some constant  $C$ . Then

$$\lambda'(x) = \frac{1}{(\tilde{\omega}' e^x)^2} \left[ A + \int_{-\infty}^x dz \tilde{\omega}'(z)^2 e^{2z} \right] \quad \text{and} \quad \lambda(x) = B + \int_0^x \frac{dy}{\tilde{\omega}'(y)^2 e^{2y}} \left[ A + \int_{-\infty}^y dz \tilde{\omega}'(z)^2 e^{2z} \right]. \quad (63)$$

We take  $B = 0$  because we want  $\Psi(0) = 0$ . Then one checks easily that one must choose  $A = 0$  because otherwise  $\Psi$  diverges at  $-\infty$ . Finally, the only possible solution to (59) is

$$\Psi(x) = \tilde{\omega}'(x) e^x \int_0^x \frac{dy}{\tilde{\omega}'(y)^2 e^{2y}} \int_{-\infty}^y dz \tilde{\omega}'(z)^2 e^{2z}, \quad (64)$$

$$= \omega'(W^{(\alpha)} + x) e^x \int_{W^{(\alpha)}}^{W^{(\alpha)} + x} \frac{dy}{\omega'(y)^2 e^{2y}} \int_{-\infty}^y dz \omega'(z)^2 e^{2z} \quad (\text{recall } \tilde{\omega}(x) = \omega(W^{(\alpha)} + x)), \quad (65)$$

$$= e^x \left[ \Phi(W^{(\alpha)} + x) - \frac{\Phi(W^{(\alpha)})}{\omega'(W^{(\alpha)})} \omega'(W^{(\alpha)} + x) \right]. \quad (66)$$

with  $\Phi$  the function (23) defined in the Theorem.

We go back to (58). Using (59), one gets

$$\partial_t s = \partial_x^2 s - \left( 2\tilde{\omega} + \delta - \eta_t + \frac{\dot{\eta}_t}{\eta_t} \right) s - \eta_t \partial_x s - \left[ \frac{\dot{\eta}_t}{\eta_t} \Psi + (\delta - \eta_t) \Psi + \eta_t \Psi' \right]. \quad (67)$$

$\Psi$  and  $\Psi'$  are bounded for  $x \leq 0$ . For large time,  $\delta(x, t)$  goes uniformly to zero.  $\eta_t$  is known [6, 8] to go to zero and, by hypothesis,  $\dot{\eta}_t/\eta_t$  also goes to 0. We conclude that there exists a positive function  $t \mapsto \epsilon_t$  which vanishes as  $t \rightarrow \infty$  and such that the term in square brackets in (67) lies for all  $x \leq 0$  in the interval  $[-\epsilon_t, \epsilon_t]$ .

The proof then goes as in Theorems 1 and 3 by using in two steps the comparison principle. For any  $t_0$ , for all  $x \leq 0$  and all  $t \geq t_0$ , one has

$$s(x, t) \leq \hat{s}(x, t) \quad \text{where} \quad \partial_t \hat{s} = \partial_x^2 \hat{s} - \left( 2\tilde{\omega} + \delta - \eta_t + \frac{\dot{\eta}_t}{\eta_t} \right) \hat{s} - \eta_t \partial_x \hat{s} + \epsilon_t, \quad \hat{s}(x, t_0) = c, \quad \hat{s}(0, t) = 0, \quad (68)$$

where  $c$  is chosen such that  $s(x, t_0) \leq c$  for all  $x \leq 0$ . It is clear that  $\hat{s}$  cannot become negative. Notice now, as before, that the big parenthesis in the equation above is larger than  $\alpha$  for  $t \geq t_0$  if  $t_0$  is chosen large enough, uniformly in  $x \leq 0$ . Then, for any non-negative function  $b_t$ ,

$$\hat{s}(x, t) \leq \bar{s}(x, t) \quad \text{where} \quad \partial_t \bar{s} = \partial_x^2 \bar{s} - \alpha \bar{s} - \eta_t \partial_x \bar{s} + \epsilon_t, \quad \bar{s}(x, t_0) = c, \quad \bar{s}(0, t) = b_t, \quad (69)$$



Choosing  $b_t$  such that  $\bar{s}$  is independent of  $x$  and solving, one obtains

$$s(x, t) \leq ce^{-\alpha t} \left( e^{\alpha t_0} + \int_{t_0}^t du \epsilon_u e^{\alpha u} \right). \quad (70)$$

From Lemma 2, the right hand side goes to zero. One bounds  $s(x, t)$  from below by a vanishing quantity in exactly the same way, therefore

$$\max_{x \leq 0} |s(x, t)| \xrightarrow{t \rightarrow \infty} 0. \quad (71)$$

Recalling that

$$u(\mu_t^{(\alpha)} + x, t) - \omega(W^{(\alpha)} + x) = \delta(x, t) = e^{-x} r(x, t) = \eta_t [e^{-x} \Psi(x) + e^{-x} s(x, t)] \quad (72)$$

and recalling the relation (66) between  $\Psi$  and  $\Phi$ , this gives the first part (22) of Theorem 2.

When  $\alpha > \frac{1}{2}$ , one can go exactly through the same steps but directly on  $\delta(x, t)$ : writing

$$\delta(x, t) = \eta_t [e^{-x} \Psi(x) + \tilde{s}(x, t)], \quad (73)$$

then  $\tilde{s} = e^{-x} s$  is solution to

$$\partial_t \tilde{s} = \partial_x^2 \tilde{s} + 2\partial_x \tilde{s} - \left( 2\tilde{\omega} - 1 + \delta + \frac{\dot{\eta}_t}{\eta_t} \right) \tilde{s} - \eta_t \partial_x \tilde{s} - \left[ \frac{\dot{\eta}_t}{\eta_t} \Psi + (\delta - \eta_t) \Psi + \eta_t \Psi' \right] e^{-x}. \quad (74)$$

One checks that the square bracket multiplied by  $e^{-x}$  is still bounded, then the parenthesis is larger than  $\alpha - \frac{1}{2}$  for  $t$  large enough and the comparison principle still applies and leads to  $\max_{x \leq 0} |\tilde{s}| \rightarrow 0$ .

The second part (24) of Theorem 2 is an easy consequence of the first part; Apply (22) to  $x_t = \mu_t^{(\beta)} - \mu_t^{(\alpha)}$ ; as  $x_t \rightarrow W^{(\beta)} - W^{(\alpha)}$ , it remains in  $[-x_0, 0]$  for  $t$  large enough and a well chosen  $x_0$ . One gets

$$\frac{\beta - \omega(W^{(\alpha)} + \mu_t^{(\beta)} - \mu_t^{(\alpha)})}{\eta_t} \xrightarrow{t \rightarrow \infty} \Phi(W^{(\beta)}) - \frac{\Phi(W^{(\alpha)})}{\omega'(W^{(\alpha)})} \omega'(W^{(\beta)}). \quad (75)$$

But

$$\omega(W^{(\alpha)} + \mu_t^{(\beta)} - \mu_t^{(\alpha)}) = \beta - (\mu_t^{(\alpha)} - \mu_t^{(\beta)} - W^{(\alpha)} + W^{(\beta)}) [\omega'(W^{(\beta)}) + o(1)], \quad (76)$$

which leads to (24).  $\square$

## 4 Numerical evidence in support of Conjecture 1

To better understand the behaviour of the  $\mu_t^{(\alpha)}$ , we made some numerical simulation. On a space-time lattice with steps  $a$  and  $b$ , we simulated the following equation:

$$h(x, t + b) = h(x, t) + \frac{b}{a^2} [h(x - a, t) + h(x + a, t) - 2h(x, t)] + b [h(x, t) - h(x, t)^2]. \quad (77)$$

We present here results for  $a = 0.1$  and  $b = 0.002$ , but we also checked other values of  $a$  and  $b$  and obtained similar results.

If one linearises (77) and looks for solutions of the form  $e^{-\gamma(x-vt)}$ , one obtains the following relation between  $v$  and  $\gamma$ :

$$v(\gamma) = \frac{1}{\gamma b} \ln \left[ 1 + \frac{b}{a^2} (e^{\gamma a} + e^{-\gamma a} - 2) + b \right], \quad (78)$$

from which one computes the critical velocity  $v_c = v(\gamma_c)$ :

$$\text{For } a = 0.1 \text{ and } b = 0.002, \quad v_c = 1.99684036732 \dots, \quad \gamma_c = 1.00074727697 \dots \quad (79)$$

With  $a$  and  $b$  small, equation (77) is close in some sense to the Fisher-KPP equation and the critical velocity and critical rate are close to 2 and 1.

We simulated the front with a step initial condition. It is expected that the relaxation of the front towards its critical travelling wave is built from what happens in a region of size  $2\sqrt{t}$  ahead of the front. It is therefore critical to have a good numerical precision for the small values of  $h$ . For this reason, the data actually stored in the computer's memory is  $\ln h$  rather than  $h$  itself. On the left of the front, each time  $\ln h$  was greater than  $-10^{-16}$ , then  $\ln h$  was set to 0. On the right of the front, the values of  $h$  were computed only up to the position  $v_c t + 10\sqrt{t} + 50$ ; the values of  $h$  on the right of that boundary were approximated to be zero. The simulation was run up to time 85 000.

At each time-step, to measure  $\mu_t^{(\alpha)}$  with a sub-lattice resolution, the computer looked at the four values of  $\ln h$  which are the closest to  $\ln \alpha$  (two above  $\ln \alpha$ , and two below  $\ln \alpha$ ). From this four values, the interpolating polynomial of degree 3 was built, and the chosen value for  $\mu_t^{(\alpha)}$  was the one for which this interpolating polynomial gave  $\ln \alpha$ .

Figure 1 shows a graph of  $\mu_t^{(1/2)} - \mu_t^{(\alpha)}$  as a function of  $1/t$  for different values of  $\alpha$  for times larger than  $10^3$ . On this scale, the data give some straight lines, suggesting that  $\mu_t^{(1/2)} - \mu_t^{(\alpha)}$  minus its large time limit is of order  $1/t$ . This suggests strongly that Conjecture 1 holds for the step initial condition and, therefore, that if there is a  $1/\sqrt{t}$  term in the asymptotic expansion of  $\mu_t^{(\alpha)}$ , then the coefficient of this term is  $\alpha$ -independent.

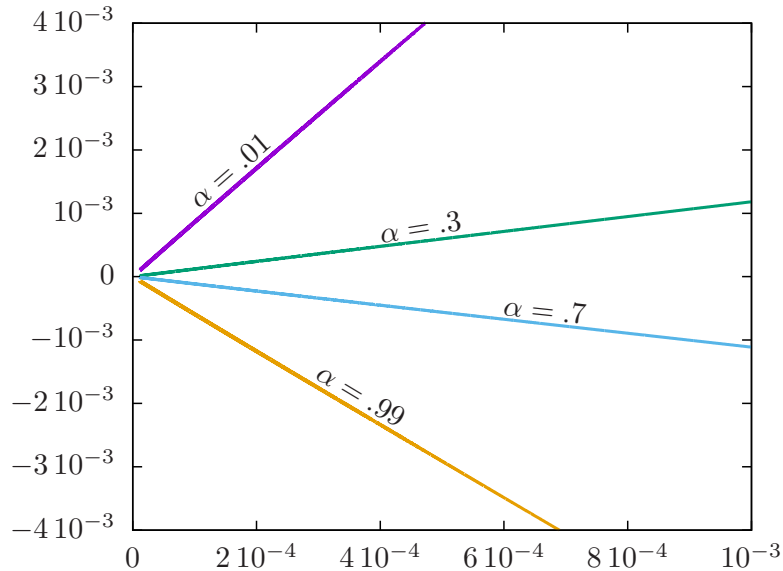


Figure 1:  $\mu_t^{(1/2)} - \mu_t^{(\alpha)} + \text{Cste}$  as a function of  $t^{-1}$ , where the constant is chosen for each  $\alpha$  so that the curves meet at the origin.

## 5 An exact expansion for a discrete solvable model

In this section, we give some arguments in support of Conjecture 2. To that end, we swap the Fisher-KPP equation we have been studying above for a model on a space lattice with continuous time which was first introduced in [3] as a front in the universality class of the Fisher-KPP equation: with  $x \in \mathbb{Z}$  and  $t \geq 0$ ,

$$\partial_t u(x, t) = \begin{cases} 0 & \text{if } u(x, t) = 1 \\ u(x, t) + au(x-1, t) & \text{if } u(x, t) < 1. \end{cases} \quad (80)$$

For this front, the function  $v(\gamma)$  is given by

$$v(\gamma) = \frac{1}{\gamma} [1 + e^\gamma], \quad (81)$$

from which one obtains  $v_c$  and  $\gamma_c$ .

As in [3], we only consider initial conditions  $u_0(x)$  such that

$$u_0(x) = 1 \text{ for } x \leq 0, \quad u_0(x) \in [0, 1) \text{ for } x \geq 1, \quad u_0(x+1) \leq u_0(x), \quad (82)$$

and we introduce, for each  $x \geq 1$ , the time  $t_x$  at which  $u(x, \cdot)$  reaches 1. It is clear that  $x \mapsto t_x$  is an increasing sequences and it was shown in [3] how to obtain the asymptotic expansion of  $t_x$  for large  $x$  up to the term  $1/\sqrt{x}$ . Pushing the same technique one step further, we obtain that for an initial condition  $u_0(x)$  that decays fast enough (see below), then

$$t_x = \frac{x}{v_c} + \frac{1}{\gamma_c v_c} \left[ \frac{3}{2} \ln x + c + \frac{d}{\sqrt{x}} + f \frac{\ln x}{x} + \mathcal{O}\left(\frac{1}{x}\right) \right] \quad \text{with } d = 3\sqrt{\pi \frac{2v_c}{v''(\gamma_c)}} \gamma_c^{-3/2}, \quad (83)$$

where  $c$  depends in a fine way on the initial condition and where  $f$  is a complicated expression involving  $v_c$ ,  $\gamma_c$ ,  $v''(\gamma_c)$  and  $v'''(\gamma_c)$ . One can check that the  $\ln x$  term in (83) is valid if  $\sum_x x u_0(x) e^{\gamma_c x} < \infty$  as in (5), the  $1/\sqrt{x}$  term is correct if  $\sum_x x^2 u_0(x) e^{\gamma_c x} < \infty$ , and the  $(\ln x)/x$  term is correct if  $\sum_x x^3 u_0(x) e^{\gamma_c x} < \infty$ . This asymptotic expansion was in [3] up to the  $1/\sqrt{x}$  term.

If one inverts formally (83), one obtains

$$x_t = v_c t - \frac{3}{2\gamma_c} \ln t + c' - \frac{d'}{\sqrt{t}} + f' \frac{\ln t}{t} + \mathcal{O}\left(\frac{1}{t}\right), \quad (84)$$

with

$$d' = \frac{d}{\gamma_c \sqrt{v_c}} = 3\sqrt{\pi \frac{2}{v''(\gamma_c)}} \gamma_c^{-5/2}, \quad f' = \frac{9}{4\gamma_c^2 v_c} - \frac{f}{\gamma_c v_c} = \frac{54 - 54 \ln 2 + 3\gamma_c \frac{v'''(\gamma_c)}{v''(\gamma_c)}}{4\gamma_c^4 v''(\gamma_c)}. \quad (85)$$

(The derivation of the value of  $f$  or  $f'$  is mechanical and tedious and of little interest. It is a simple application of the techniques explained in [3] pushed one step further.)

Conjecture 2 relies simply on the idea that the asymptotic expansion of the  $\mu_t^{(\alpha)}$  in the Fisher-KPP equation is also given by (84) with an  $\alpha$ -dependent constant  $c'$ , as in (10). However, from Conjecture 1, the coefficient of the  $(\ln t)/t$  term should be independent of  $\alpha$ . If one assumes that (85) also holds for the Fisher-KPP, one obtains, with  $\gamma_c = 1$ ,  $v''(\gamma_c) = 2$  and  $v'''(\gamma_c) = -6$ , that

$$\mu_t^{(\alpha)} = 2t - \frac{3}{2} \ln t + C + W^{(\alpha)} - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6 \ln 2) \frac{\ln t}{t} + \mathcal{O}\left(\frac{1}{t}\right), \quad (86)$$

where one recognizes in particular the Ebert and van Saarloos term [4].

We tried to see if we could see this  $(\ln t)/t$  in the numerical simulations we discussed in Section 4. To do this, we first subtracted all the known terms in  $\mu_t^{(\alpha)}$  and computed

$$\delta_t^{(\alpha)} = \mu_t^{(\alpha)} - v_c t + \frac{3}{2\gamma_c} \ln t + \frac{d'}{\sqrt{t}}. \quad (87)$$

(Of course, we used  $v_c$  and  $\gamma_c$  given by (79). Similarly, the value used for  $d'$  is not  $3\sqrt{\pi}$  but the value given in (85).) We then fitted (using **gnuplot**) the  $\delta_t^{(\alpha)}$  to extract the parameters we needed. Performing this fit is difficult: we fit against asymptotic expansions, so we need to consider large times only. On the other hand, if one fits over too narrow an interval, it is very hard to distinguish between  $(\ln t)/t$  and  $1/t$ . To overcome these difficulties, it seemed necessary to fit over a large time interval (to be able to distinguish a  $\ln t$  from a constant) and to include more terms in the expansion to gain in accuracy at smaller times. To allow the reader to better evaluate our numerical results, we present results for several fits: we used the following candidates for fitting the data:

$$\begin{aligned} \text{(a)} \quad \delta_t &= C + \frac{f' \ln t + g}{t}, & \text{(b)} \quad \delta_t &= C + \frac{f' \ln t + g}{t} + \frac{h \ln t + i}{t^{3/2}}, \\ \text{(c)} \quad \delta_t &= C + \frac{f' \ln t + g}{t} + \frac{h \ln t + i}{t^{3/2}} + \frac{j \ln t + k}{t^2}, \end{aligned} \quad (88)$$

over different ranges of  $t$ . The values of  $f'$  extracted from the fits are presented in Table 1. When using function (a), these values depend a lot on the chosen range. This is because the effects of smaller terms in the expansion is not negligible enough for the values of  $t$  that we could reach. Function (b) seems to suffer a little bit from this effect, but to a much lesser extent. Function (c) leads to a remarkable

	on [100,85000]	on [1000,85000]	on [10000,85000]
With function (a)	1.642	1.355	1.164
	1.639	1.302	1.131
	1.630	1.288	1.123
	1.619	1.274	1.114
	1.501	1.177	1.060
With function (b)	0.805	0.907	0.933
	0.896	0.928	0.938
	0.912	0.932	0.939
	0.926	0.937	0.941
	0.979	0.958	0.947
With function (c)	0.938	0.945	0.937
	0.945	0.944	0.936
	0.945	0.944	0.938
	0.944	0.944	0.938
	0.935	0.943	0.935

Table 1: The value of  $f'$  when fitting the  $\delta_t^{(\alpha)}$  against the functions in (88) over three time ranges. In each cell, the five values correspond from top to bottom to  $\alpha = 0.01$ ,  $\alpha = 0.3$ ,  $\alpha = 0.5$ ,  $\alpha = 0.7$  and  $\alpha = 0.99$ .

uniformity of values for  $f'$ . According to (85), the value of  $f'$  should be  $0.948\dots$ , which is in quite good agreement with the fitted values. (For the Fisher-KPP, the value for  $f'$  in (86) is  $0.946\dots$ ).

On Figure 2,  $t(\delta_t^{(\alpha)} - C^{(\alpha)})$  is plotted as a function of  $t$  on a log-lin scale, using for  $C^{(\alpha)}$  the value obtained from the fit with function (c) over  $[1000, 85000]$ . The curves seem to have an asymptote, which would indicate that

- The Ebert-van Saarloos correction in  $1/\sqrt{t}$  is indeed the first vanishing term in the asymptotic expansion of  $\mu_t^{(\alpha)}$ , with the predicted coefficient. (If the prefactor were wrong, the curves in Figure 2 would blow up exponentially fast in the log-lin scale.)
- After the Ebert-van Saarloos correction, the next term in the asymptotic expansion of  $\mu_t^{(\alpha)}$  seems indeed to be a  $(\ln t)/t$ .
- The prefactor of the  $(\ln t)/t$ , which is given by the slope of the asymptote of the curves in Figure 2, is possibly equal to the  $\alpha$ -independent value predicted in (85).

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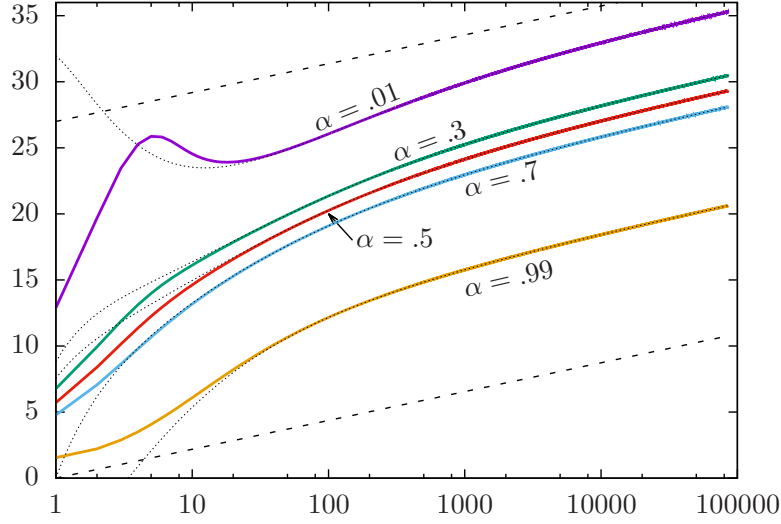


Figure 2:  $t(\delta_t^{(\alpha)} - C^{(\alpha)})$  as a function of  $t$ , on a log-lin scale. The value of  $C^{(\alpha)}$  was obtained from the fit with function (c) over  $[1000, 85000]$ . The small dotted lines show, for each  $\alpha$  the result of the fit. The two straight dashed lines are  $0.946 \ln t + \text{Cste}$ .

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